NOTES ON THE MOTIVIC MCKAY CORRESPONDENCE FOR THE GROUP SCHEME α_p

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Abstract. We formulate a conjecture on the motivic McKay correspondence for the group scheme α_p in characteristic $p > 0$ and give a few evidences. The conjecture especially claims that there would be a close relation between quotient varieties by α_p and ones by the cyclic group of order p.

1. Introduction

The motivic McKay correspondence was established by Batyrev [1] and Denef–Loeser [3] in characteristic zero. A version of this theory says that given a linear action of a finite group G on an affine space $\mathbb{A}_k^d$ without pseudo-reflection, we can express the motivic stringy invariant $M_{\mathrm{st}}(\mathbb{A}_k^d/G)$ of the quotient variety $\mathbb{A}_k^d/G$ as a sum of the form $\sum_{g\in \mathrm{Conj}(G)} \mathbb{L}^{a(g)}$, where $\mathrm{Conj}(G)$ is the set of conjugacy classes of G and a is a function on $\mathrm{Conj}(G)$ with values in $\frac{1}{\sharp G}\mathbb{Z}$. This can be generalized to the tame case in characteristic $p > 0$ (the case $p \nmid \sharp G$) without essential change (see [9]). After studying the case of the cyclic group of order p, the second author formulated a conjectural generalization to the wild case (the case $p \mid \sharp G$) (see [10, 11]). In this conjecture, the sum $\sum_{g\in \mathrm{Conj}(G)} \mathbb{L}^{a(g)}$ is replaced with an integral of the form $\int_{\Delta_G} \mathbb{L}^{a(g)}$, where Δ_G is the moduli space of G-torsors over the punctured formal disk $\operatorname{Spec} k((t))$ and a is a $\frac{1}{\sharp G}\mathbb{Z}$-valued function on it.

The aim of this paper is to discuss the case where G is the group scheme α_p rather than a genuine finite group, as a first step towards further generalization to arbitrary finite group schemes. We will formulate a conjectural expression (Conjecture 4.3) for $M_{\mathrm{st}}(\mathbb{A}_k^d/G)$ again of the form $\int_{\Delta_G} \mathbb{L}^{a(g)}$ under the condition $D_{\mathbf{d}} \geq 2$ (for the definition of $D_{\mathbf{d}}$, see Section 4). But here we have a new phenomenon: the moduli space Δ_G in this case is an ind-pro-limit of finite dimensional spaces rather than an ind-limit as in the case of genuine finite groups. Therefore we need to define a motivic measure on Δ_G in terms of truncation maps as we do for the arc space. Our conjecture also indicates a close relation between the case of $G = \alpha_p$ and the case of the cyclic group $H := \mathbb{Z}/p\mathbb{Z}$ of order p. There exists a one-to-one correspondence between G-representations and H-representations. The conjecture says that if $\mathbb{A}_k^d/G$ and $\mathbb{A}_k^d/H$ are quotient varieties associated to G and H-representations corresponding to each other, then they have equal motivic stringy invariant. We will give a few evidences for this conjecture. Note that Hiroyuki Ito [6] earlier observed a similarity between surface quotient singularities by (non-linear) G-actions and H-actions. This was an inspiration for our conjecture.

The paper is organized as follows. In Section 2, we describe the moduli spaces Δ_G and Δ_H. In Section 3, we define motivic measures on these spaces. In Section 4, we formulate our main conjecture. In Section 5, we give two examples supporting the equality of stringy invariants of $\mathbb{A}_k^d/G$ and $\mathbb{A}_k^d/H$. In Section 6, we discuss the case where G acts on $\mathbb{A}_k^2$, that is, the case $D_{\mathbf{d}} = 1$, as a toy model and show the change of variables formula for the quotient map $\mathbb{A}_k^2 \to \mathbb{A}_k^2/G$. This would be viewed as a supporting evidence for our conjecture that $M_{\mathrm{st}}(\mathbb{A}_k^d/G)$ is expressed as $\int_{\Delta_G} \mathbb{L}^{a(g)}$ when $D_{\mathbf{d}} \geq 2$. In Appendix, we briefly recall the representation theory of α_p in terms of relations to nilpotent endomorphisms and derivations.

In what follows, we work over an algebraically closed field k of characteristic $p > 0$. We always denote by G the group scheme α_p and by H the cyclic group of order p. We write the coordinate ring of α_p as $k[\epsilon] = k[x]/(x^p)$ with ϵ the image of x in this quotient ring.

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2. Moduli of G and H-torsors over the punctured formal disk.

The group scheme G fits into the exact sequence

$$0 \to G \to \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \to 0,$$

where F is the Frobenius map. Therefore the G-torsors over $\operatorname{Spec} k((t))$ are parameterized by $k((t))/F(k((t)))$. Let

$$\Delta_G := \left\{ \sum_{i \in \mathbb{Z};\, p \nmid i} c_i t^i \mid c_i \in k \right\} \subset k((t)),$$

the set of Laurent power series having only terms of degree prime to p. The inclusion map $\Delta_G \to k((t))$ induces a bijection $\Delta_G \to k((t))/F(k((t)))$. Thus we regard Δ_G as the "moduli space" of G-torsors over $\operatorname{Spec} k((t))$. The G-torsor corresponding to a Laurent power series $f \in \Delta_G$ is $\operatorname{Spec} k((t))[z]/(z^p - f)$ and the action of α_p is defined so that the associated coaction is the $k((t))$-algebra homomorphism

$$\frac{k((t))[z]}{(z^p - f)} \to \frac{k((t))[z]}{(z^p - f)}[\epsilon],\ z \mapsto z + \epsilon.$$

We can make a similar construction for H-torsors. Let $\wp\colon k((t)) \to k((t))$ be the Artin-Schreier map given by $\wp(f) = f^p - f$ and let

$$\Delta_H := \left\{ \sum_{i \in \mathbb{Z};\, i<0,\, p \nmid i} c_i t^i \mid c_i \in k \right\} \subset k((t)),$$

the set of Laurent polynomials having only terms of *negative* degree prime to p. Then the H-torsors are parameterized by $k((t))/\wp(k((t)))$ and the composite map $\Delta_H \hookrightarrow k((t)) \twoheadrightarrow k((t))/\wp(k((t)))$ is bijective. Thus we regard Δ_H as the "moduli space" of H-torsors over $\operatorname{Spec} k((t))$. The H-torsor corresponding to $f \in \Delta_H$ is $\operatorname{Spec} k((t))[z]/(z^p - z - f)$ where a generator of H acts by $z \mapsto z + 1$.

Remark 2.1. Constructing the true moduli space (stack) which represents a relevant moduli functor is a more difficult problem. However, the above ad hoc version would be sufficient for our application to motivic integration, since we work over an algebraically closed field. When k is algebraically closed, the coarse moduli space for Δ_H was constructed by Harbater [4]. The fine moduli stack for Δ_H over an arbitrary field of characteristic $p > 0$ was constructed by the authors [7]. The moduli space Δ_G has not been seriously studied yet, as far as the authors know.

3. Motivic measures on Δ_G and Δ_H

For a positive integer j prime to p, the set $\Delta_H^{\geq -j} := \{f \in \Delta_H \mid \mathrm{ord}(f) \geq -j\}$ is the affine space $\mathbb{A}_k^{j-\lfloor j/p \rfloor}$. Thus Δ_H is the union of affine spaces $\Delta_H^{\geq -j}$. We say that a subset C of Δ_H is *constructible* if C is a constructible subset of some $\Delta_H^{\geq -j}$. We define the *motivic measure* μ_H on Δ_H by $\mu_H(C) := [C]$ say in $\hat{\mathcal{M}}'$, a version of the complete Grothendieck ring of varieties used in [10]. (In this note, we do not discuss what additional relation would be really necessary to put on the complete Grothendieck ring for the McKay correspondence in the case of the group scheme G. This should be clarified in a future study.)

For $n \in \mathbb{Z}$, let

$$\tau_n \colon \Delta_G \to \Delta_{G,n} := \frac{\Delta_G}{\Delta_G \cap t^{np} k[[t]]}$$

be the quotient map, which truncates the terms of degrees $\geq np$. We often identify $\Delta_{G,0}$ with Δ_H through the natural bijection $\Delta_H \hookrightarrow \Delta_G \to \Delta_{G,0}$ and τ_0 with a map $\Delta_G \to \Delta_H$. When $n \geq 0$, $\Delta_{G,n}$ is the product of Δ_H and an affine space $\mathbb{A}_k^{n(p-1)}$. We say that a subset $C \subset \Delta_G$ is a *cylinder of level* n if $\tau_n(C)$ is a constructible subset of $\Delta_{G,n}$ and $C = \tau_n^{-1}(\tau_n(C))$. For a cylinder $C \subset \Delta_G$ of level n, we define its measure as

$$\mu_G(C) := [\tau_n(C)]\mathbb{L}^{-n(p-1)} \in \hat{\mathcal{M}}'.$$

Since the natural map $\Delta_{G,n+1} \to \Delta_{G,n}$ is the trivial $\mathbb{A}_k^{p-1}$-fibration, the element $[\tau_n(C)]\mathbb{L}^{-n(p-1)}$ is independent of the choice of a sufficiently large n. For instance, $\Delta_G^{\geq 0} := \{f \in \Delta_G \mid \mathrm{ord}(f)\}$ is a cylinder of level zero such that $\tau_0(\Delta_G^{\geq 0})$ is a singleton. Note that this set contains $0 \in k((t))$, according to the convention $\mathrm{ord}(0) = +\infty$. The measure of $\Delta_G^{\geq 0}$ is

$$\mu_G(\Delta_G^{\geq 0}) = [1\,\mathrm{pt}] = 1.$$

4. A conjecture on the McKay correspondence for linear G-actions

To a sequence of integers, $\mathbf{d} = (d_1, d_2, \ldots, d_l)$ such that $1 \leq d_\lambda \leq p$ and $d_\lambda \geq d_{\lambda+1}$, we associate the linear representation W of H over k that have d_λ-dimensional indecomposables as direct summands. Namely a generator of H acts on the vector space $W = k^d$, $d := |\mathbf{d}| = \sum_{\lambda=1}^{l} d_\lambda$ by a matrix whose Jordan normal form has Jordan blocks of sizes d_λ with diagonal entries 1. The map $\mathbf{d} \mapsto W$ gives a one-to-one correspondence between sequences $\mathbf{d}$ of integers as above and isomorphism classes of H-represention over k.

Similarly, to a sequence $\mathbf{d}$ as above, we associate also the linear representation V of G over k that have d_λ-dimensional indecomposables as direct summands. If ρ denotes the nilpotent linear endomorphism of $k[x_1, \ldots, x_d]$ defined by Jordan blocks of sizes d_λ with diagonal entries 0, then the map

$$k[x_1, \ldots, x_d] \to k[x_1, \ldots, x_d][\epsilon],\ f \mapsto \sum_{i=0}^{p-1} \frac{\rho^i(f)}{i!}\epsilon^i$$

defines the linear G-action on $\mathbb{A}_k^d$. For details, see Appendix. In particular, through sequences $\mathbf{d}$, we get a one-to-one correspondence between G-representations and H-representations. We expect that the quotient varieties V/G and W/H for the same $\mathbf{d}$ would be very similar in the sense we will make more precise.

Let us fix a sequence $\mathbf{d}$ as above. Following [10], for a positive integer j prime to p, we define

$$\operatorname{sht}(j) := \sum_{\lambda=1}^{l} \sum_{i=1}^{d_\lambda - 1} \left\lfloor \frac{ij}{p} \right\rfloor .$$

We define a function $\operatorname{sht}\colon \Delta_H \to \mathbb{Z}$ by

$$\operatorname{sht}(f) := \begin{cases} \operatorname{sht}(-\operatorname{ord}(f)) & (f \neq 0) \\ 0 & (f = 0). \end{cases}$$

We define another function sht' on Δ_H by

$$\operatorname{sht}'(f) := \begin{cases} \operatorname{sht}(f) - l & (f \neq 0) \\ -d & (f = 0). \end{cases}$$

Fibers of these functions are constructible subsets. In general, for a function $u\colon \Delta_H \to \mathbb{Z}$ with constructible fibers, we define the motivic integral

$$\int_{\Delta_H} \mathbb{L}^u \, d\mu_H := \sum_{i \in \mathbb{Z}} [u^{-1}(i)] \mathbb{L}^i \in \hat{\mathcal{M}}',$$

provided that the last infinite sum converges in $\hat{\mathcal{M}}'$. When it diverges, we formally put $\int_{\Delta_H} \mathbb{L}^f \, d\mu_H := \infty$.

We define a numerical invariant $D_{\mathbf{d}} := \sum_{\lambda=1}^{l} (d_\lambda - 1) d_\lambda / 2$, which we also think of as invariants of representations $V_{\mathbf{d}}$ and $W_{\mathbf{d}}$. Integrals

$$\int_{\Delta_H} \mathbb{L}^{-\operatorname{sht}} \, d\mu_H, \quad \int_{\Delta_H} \mathbb{L}^{-\operatorname{sht}'} \, d\mu_H$$

converge exactly when $D_{\mathbf{d}} \geq p$ (for details of computation, see [10]). If $D_{\mathbf{d}} = 0$, then the corresponding H-action is trivial. If $D_{\mathbf{d}} = 1$, then $\mathbf{d} = (2, 1, \ldots, 1)$, $W/H \cong \mathbb{A}_k^d$ and the quotient map $W \to W/H$ has ramification locus of codimension one. Therefore the case $D_{\mathbf{d}} \geq 2$ is of our main interest, although the case $D_{\mathbf{d}} = 1$ will be discussed in Section 6 as a toy case.

The quotient variety W/H is factorial [2, Th. 3.8.1], in particular, has the invertible canonical sheaf $\omega_{W/H}$. The *ω-Jacobian ideal sheaf* $\mathcal{J} \subset \mathcal{O}_{W/H}$ is defined by $\mathcal{J}\omega_{W/H} := \operatorname{Im}(\bigwedge^{|\mathbf{d}|} \Omega_{W/H} \to \omega_{W/H})$. Let $o \in W/H$ be the image of the origin of W, $J_\infty(W/H)$ the arc space of W/H and $J_\infty(W/H)_o$ the preimage of o by the natural map $J_\infty(W/H) \to W/H$. The *motivic stringy invariant* (resp. the *motivic stringy invariant at o*) of W/H is defined to be

$$M_{\mathrm{st}}(W/H) := \int_{J_\infty(W/H)} \mathbb{L}^{\operatorname{ord} \mathcal{J}} \, d\mu_{W/H}$$

$$\left(\text{resp. } M_{\mathrm{st}}(W/H)_o := \int_{J_\infty(W/H)_o} \mathbb{L}^{\operatorname{ord} \mathcal{J}} \, d\mu_{W/H} \right).$$

If there exists a crepant resolution $\phi\colon Y \to W/H$, then we have $M_{\mathrm{st}}(W/H) = [Y]$ and $M_{\mathrm{st}}(W/H)_o = [\phi^{-1}(o)]$.

The following theorem proved in [10] can be considered as the motivic McKay correspondence for linear H-actions:

Theorem 4.1. *If $D_{\mathbf{d}} \geq 2$, we have the following equalities in $\hat{\mathcal{M}}' \cup \{\infty\}$,*

$$M_{\mathrm{st}}(W/H)_o = \int_{\Delta_H} \mathbb{L}^{-\mathrm{sht}}\, d\mu_H, \quad M_{\mathrm{st}}(W/H) = \int_{\Delta_H} \mathbb{L}^{-\mathrm{sht}'}\, d\mu_H.$$

Since the convergence of $M_{\mathrm{st}}(W/H)$ (or $M_{\mathrm{st}}(M/H)_o$) is equivalent to that W/H has only canonical singularities, this theorem in particular implies that W/H has canonical singularities if and only if $D_{\mathbf{d}} \geq p$ (see [8]).

In the rest of this section, we will formulate a conjecture for V/G similar to this theorem. For this purpose, we extend functions $\mathrm{sht}, \mathrm{sht}' \colon \Delta_H \to \mathbb{Z}$ to functions on Δ_G by

$$\mathrm{sht}(f) = \mathrm{sht}(\tau_0(f)) = \begin{cases} \mathrm{sht}(-\mathrm{ord}(f)) & (\mathrm{ord}(f) < 0) \\ \mathrm{sht}(0) & (\mathrm{ord}(f) \geq 0) \end{cases},$$

$$\mathrm{sht}'(f) := \mathrm{sht}'(\tau_0(f)) = \begin{cases} \mathrm{sht}'(-\mathrm{ord}(f)) & (\mathrm{ord}(f) < 0) \\ \mathrm{sht}'(0) & (\mathrm{ord}(f) \geq 0) \end{cases}.$$

Each fiber of the extended functions sht or sht' is a cylinder of level zero and

$$\tau_0(\mathrm{sht}^{-1}(i)) = (\mathrm{sht}|_{\Delta_H})^{-1}(i), \quad \tau_0((\mathrm{sht}')^{-1}(i)) = \left(\mathrm{sht}'|_{\Delta_H}\right)^{-1}(i). \tag{4.1}$$

For a function $u \colon \Delta_G \to \mathbb{Z}$ whose fibers are cylinders, we define

$$\int_{\Delta_G} \mathbb{L}^u\, d\mu_G := \sum_{i \in \mathbb{Z}} \mu_G(u^{-1}(i))\mathbb{L}^i.$$

From (4.1), we have

$$\int_{\Delta_G} \mathbb{L}^{-\mathrm{sht}}\, d\mu_G = \int_{\Delta_H} \mathbb{L}^{-\mathrm{sht}}\, d\mu_H, \quad \int_{\Delta_G} \mathbb{L}^{-\mathrm{sht}'}\, d\mu_H = \int_{\Delta_H} \mathbb{L}^{-\mathrm{sht}'}\, d\mu_H.$$

Lemma 4.2. *The scheme V/G is of finite type over k and factorial.*

Proof. Let $k[V]$ and $k[V/G]$ be the coordinate rings of V and V/G respectively, so that $k[V/G] = k[V]^D$. In particular $k[V]^p \subseteq k[V/G]$, which easily implies that $V \longrightarrow V/G$ is a homeomorphism. Moreover $V/G \longrightarrow V$ is finite, which implies that V/G is of finite type over k because k is perfect.

Let P be a prime of height one of $k[V/G]$. We must show that P is principal. We have that P is the restriction of a height one prime ideal of $k[V]$, which is therefore generated by an irreducible polynomial f. If $f \mid D(f)$ then, since $\deg(D(f)) \leq \deg(f)$, we have $D(f) = cf$ for some $c \in k$. In particular $0 = D^p(f) = c^p f$ so that $c = 0$ and $D(f) = 0$. In this case it follows easily that $P = k[V/G] \cap (fk[V]) = fk[V/G]$.

So assume that $f \nmid D(f)$. We claim that $P = f^p k[V/G]$. Let $x \in P - \{0\}$, that is $x = hf^l$ with $l > 0$, h coprime with f and $D(x) = 0$. We have

$$0 = D(x) = D(h)f^l + hlf^{l-1}D(f) \implies p \mid l \text{ and } D(h) = 0$$

So $x = h(f^p)^{l/p} \in f^p k[V/G]$. □

Thanks to this lemma, we can define the ω-Jacobian ideal on V/G similarly to the case of W/H. In turn, we can define $M_{\mathrm{st}}(V/G), M_{\mathrm{st}}(V/G)_o$. The following is our main conjecture.

Conjecture 4.3. *If $D_{\mathbf{d}} \geq 2$, we have the following equalities in $\hat{\mathcal{M}}' \cup \{\infty\}$,*

$$M_{\mathrm{st}}(V/G)_o = \int_{\Delta_G} \mathbb{L}^{-\mathrm{sht}_{\mathbf{d}}} \, d\mu_G \, (= M_{\mathrm{st}}(W/H)_o) \,,$$
$$M_{\mathrm{st}}(V/G) = \int_{\Delta_G} \mathbb{L}^{-\mathrm{sht}'_{\mathbf{d}}} \, d\mu_G \, (= M_{\mathrm{st}}(W/H)) \,.$$

5. Two Examples

In this section, we see two examples supporting the conjecture that $M_{\mathrm{st}}(V/G) = M_{\mathrm{st}}(W/H)$ and $M_{\mathrm{st}}(V/G)_o = M_{\mathrm{st}}(W/H)_o$.

5.1. We first consider the case $\mathbf{d} = (3)$, supposing $p \geq 3$. If $p = 3$, then W/H has a crepant resolution $\phi\colon U \to W/H$ such that

$$M_{\mathrm{st}}(W/H) = [U] = \mathbb{L}^3 + 2\mathbb{L}^2,$$
$$M_{\mathrm{st}}(W/H)_o = [\phi^{-1}(o)] = 2\mathbb{L} + 1.$$

If $p > 3$, then W/H is not log canonical, in particular, $M_{\mathrm{st}}(W/H) = M_{\mathrm{st}}(W/H)_o = \infty$. See [10, Example 6.23].

As for the G-action on V, the corresponding derivation D on the coordinate ring $k[x, y, z]$ is given by $D(x) = 0$, $D(y) = x$, $D(z) = y$. We can compute

$$k[V/G] = k[x, y^p, z^p, y^2 - 2xz] \cong k[X, Y, Z, W]/(Y^2 - W^p - X^p Z),$$

where X, Y, Z, W correspond to $2x, y^p, z^p, y^2 - 2xz$ respectively. By simple computation of blowups, we can easily see that if $p = 3$, then V/G has a crepant resolution. Using this resolution, we see that $M_{\mathrm{st}}(V/G) = \mathbb{L}^3 + 2\mathbb{L}^2$ and $M_{\mathrm{st}}(V/G)_o = 2\mathbb{L} + 1$. We also see that if $p > 3$, then V/G is not canonical and $M_{\mathrm{st}}(V/G) = \infty$. More details of computation are as follows.

Let us compute a (partial) resolution of $U_0 := V/G$. The singular locus $U_{0,\mathrm{sing}}$ of U_0 is the affine line defined by $X = Y = W = 0$. Let $U_1 \to U_0$ be the blowup of U_0 along $U_{0,\mathrm{sing}}$. This is a crepant morphism and the exceptional locus is the trivial $\mathbb{P}^1$-bundle over $U_{0,\mathrm{sing}}$.

When $p = 3$, then $U_{1,\mathrm{sing}}$ is again an affine line and $U_{1,\mathrm{sing}}$ has an affine open neighborhood isomorphic to $\operatorname{Spec} k[X, Y, Z]/(Y^2 + XZ) \times \mathbb{A}^1_k$. Therefore the blowup $U_2 \to U_1$ along $U_{1,\mathrm{sing}}$ is a crepant resolution and its exceptional locus is again the trivial $\mathbb{P}^1$-bundle over $U_{1,\mathrm{sing}}$. Let $\phi\colon U_2 \to U_0$ be the natural morphism, which is a crepant resolution. Then, the above computation shows that

$$M_{\mathrm{st}}(V/G) = [U_2] = \mathbb{L}^3 + 2\mathbb{L}^2,$$
$$M_{\mathrm{st}}(V/G)_o = [\phi^{-1}(o)] = 2\mathbb{L} + 1.$$

When $p > 3$, we claim that U_1 is not log canonical, and neither is $U_0 = V/G$, since $U_1 \to U_0$ is crepant. By an explicit computation, we see that U_1 has an affine open subset U'_1 which is isomorphic to a hypersurface defined by $Y^2 - X^{p-2}W^p - X^{p-2}Z = 0$. Its singular locus $U'_{1,\mathrm{sing}}$ is defined by $X = Y = 0$. Let $U_2 \to U'_1$ be the blowup along $U'_{1,\mathrm{sing}}$ and $B \to \mathbb{A}^4_k$ the blowup of the ambient affine space along the same locus $U'_{1,\mathrm{sing}}$ so that U_2 is a closed subset of B. Let $E \subset B$ be the exceptional divisor of $B \to \mathbb{A}^4_k$. We see

$$K_{U_2/U'_1} = -E|_{U_2}.$$

If $p = 5$, then U_2 is normal and $-E|_{U_2} = -2E'$ for a prime divisor E'. Thus E' has discrepancy -2 over U_1, hence U_1 is not log canonical. If $p > 5$, then U_2 is not normal. We similarly take the blowup $\psi\colon U_3 \to U_2'$ of an affine open subset $U_2' \subset U_2$ such that $K_{U_3/U_2'} = -F|_{U_3}$ and $\psi^*(E|_{U_2}) = F|_{U_3}$, where F is the exceptional divisor of the blowup of the ambient affine space. We conclude that $K_{U_3/U_1'} = -2F|_{U_3}$. This shows that U_1 is not log canonical. Thus V/G is not log canonical and we have $M_{\mathrm{st}}(V/G) = \infty$. Let $T \to V/G$ be a log resolution on which the above exceptional divisor with discrepancy < -1 appears. This exceptional divisor on T surjects onto the singular locus of V/G. This shows that $M_{\mathrm{st}}(V/G)_o = \infty$.

Remark 5.1. When $p = 3$, the quotient singularities above by H and G are the same as the two hypersurface singularities in characteristic three in [5, Th. 3] up to suitable coordinate transforms.

5.2. Next we consider the case $p = 2$ and $\mathbf{d} = (2, 2)$. Then W/H is the symmetric product of two copies of $\mathbb{A}^2_k$. It has a crepant resolution $\phi\colon U \to W/H$ constructed simply by blowing up the singular locus once. This resolution coincides with the Hilbert scheme of two points of $\mathbb{A}^2_k$, usually denoted by $\mathrm{Hilb}^2(\mathbb{A}^2_k)$. From an explicit description, we know that

$$M_{\mathrm{st}}(W/H) = [U] = \mathbb{L}^4 + \mathbb{L}^3,$$
$$M_{\mathrm{st}}(W/H)_o = [\phi^{-1}(o)] = \mathbb{L} + 1.$$

The corresponding derivation on the polynomial ring $k[x_0, y_0, x_1, y_1]$ is given by $D(x_i) = 0$, $D(y_i) = x_i$. The coordinate ring of V/G is

$$k[x_0, y_0^2, x_1, y_1^2, x_0y_1 + x_1y_0] \cong k[V, W, X, Y, Z]/(Z^2 + V^2Y + X^2W).$$

The singular locus of V/G is defined by $X = V = Z = 0$ and isomorphic to an affine plane. Let $\phi\colon U \to V/G$ be the blowup along the singular locus. This is a crepant resolution such that the exceptional locus is the trivial $\mathbb{P}^1$-bundle over $(V/G)_{\mathrm{sing}} \cong \mathbb{A}^2_k$. Thus $[U] = \mathbb{L}^4 + \mathbb{L}^3$ and $[\phi^{-1}(o)] = \mathbb{L} + 1$.

6. The change of variables in dimension two

In this section, we present some computation supporting the equalities $M_{\mathrm{st}}(V/G)_o = \int_{\Delta_G} \mathbb{L}^{-\mathrm{sht}_{\mathbf{d}}}\, d\mu_G$ and $M_{\mathrm{st}}(V/G) = \int_{\Delta_G} \mathbb{L}^{-\mathrm{sht}'_{\mathbf{d}}}\, d\mu_G$ in Conjecture 4.3. However, when $D_{\mathbf{d}} \geq 2$, V/G and W/H have singularities, which make analysis more difficult. Therefore we consider the case $D_{\mathbf{d}} = 1$ as a toy model. We will use jet schemes and the theory of integration above those spaces. For generalities see [10, Section 4].

Then $\mathbf{d}$ is of the form $(2, 1, \dots, 1)$, but it is enough to consider the special case $\mathbf{d} = (2)$, because there is no essential difference in the general case. When $\mathbf{d} = (2)$, if we write $W = \operatorname{Spec} k[x, y]$ and let a generator of H act on it by $y \mapsto y$, $x \mapsto x + y$, then $W/H = \operatorname{Spec} k[x^p - xy^{p-1}, y] \cong \mathbb{A}^2_k$. The quotient map $W \to W/H$ is ramified along the divisor y, in particular, the map is not crepant. Because of this, we do not have the equalities in Conjecture 4.3 in this case. Instead we have

$$\mathbb{L}^2 = M_{\mathrm{st}}(W/H) = \int_{J_\infty(W/H)} 1\, d\mu_{W/G} = \int_{\mathcal{J}_\infty \mathcal{W}} \mathbb{L}^{-\mathrm{ord}(y^{p-1}) - \mathfrak{s}}\, d\mu_{\mathcal{W}},$$
$$1 = M_{\mathrm{st}}(W/H)_o = \int_{(J_\infty(W/H))_o} 1\, d\mu_{W/G} = \int_{(\mathcal{J}_\infty \mathcal{W})_o} \mathbb{L}^{-\mathrm{ord}(y^{p-1}) - \mathfrak{s}}\, d\mu_{\mathcal{W}}$$

the last integral of which we will explain now. The domain of integral, $\mathcal{J}_\infty \mathcal{W}$, is the space of twisted arcs of the quotient stack $\mathcal{W} = [W/H]$. The use of the stack $\mathcal{W}$ is only for this conventional notation and not really necessary. We can describe this space as

$$\mathcal{J}_\infty \mathcal{W} := \bigsqcup_{f \in \Delta_H} \mathrm{Hom}^H(\operatorname{Spec} \mathcal{O}_f, W)/H,$$

where $\operatorname{Spec} \mathcal{O}_f$ is the normalization of $\operatorname{Spec} k[[t]]$ in the H-torsor over $\operatorname{Spec} k((t))$ corresponding to f, $\mathrm{Hom}^H(-,-)$ is the set of H-equivariant morphisms. The $(\mathcal{J}_\infty \mathcal{W})_o$ is the subset of $\mathcal{J}_\infty \mathcal{W}$ consisting of H-orbits of H-equivariant maps $\operatorname{Spec} \mathcal{O}_f \to W$ sending the closed point(s) onto the origin of W, and $(J_\infty(W/H))_o$ is the set of arcs $\operatorname{Spec} k[[t]] \to W/G$ sending the closed point to o. We can define a motivic measure on $\mathcal{J}_\infty \mathcal{W}$ and there exists a natural map $\mathcal{J}_\infty \mathcal{W} \to J_\infty(W/H)$ which is almost bijective (bijective outside measure zero subsets) and induces an almost bijection $(\mathcal{J}_\infty \mathcal{W})_o \to (J_\infty(W/H))_o$. The $\mathrm{ord}(y^{p-1})$ is the function on $\mathcal{J}_\infty \mathcal{W}$ assigning orders of y^{p-1} along twisted arcs. Note that the ideal $(y^{p-1}) \subset k[x,y]$ is the Jacobian ideal of the map $W \to W/H$. Finally $\mathfrak{s}$ is the composition of the natural map $\mathcal{J}_\infty \mathcal{W} \to \Delta_H$ and $\mathrm{sht}\colon \Delta_H \to \mathbb{Z}$. The equality $\int_{J_\infty(W/H)} 1 \, d\mu_{W/H} = \int_{\mathcal{J}_\infty \mathcal{W}} \mathbb{L}^{-\mathrm{ord}(y^{p-1})-\mathfrak{s}} \, d\mu_{\mathcal{W}}$ can be thought of as the change of variables formula for the map $\mathcal{J}_\infty \mathcal{W} \to J_\infty(W/H)$. More generally, for a measurable function $F\colon C \to \mathbb{Z}$ on a subset $C \subset J_\infty(W/G)$, if $\phi\colon \mathcal{J}_\infty \mathcal{W} \to J_\infty(W/G)$ denotes the natural map, then

$$\int_C \mathbb{L}^F \, d\mu_{W/H} = \int_{\phi^{-1}(C)} \mathbb{L}^{F\circ\phi-\mathrm{ord}(y^{p-1})-\mathfrak{s}} \, d\mu_{\mathcal{W}}. \tag{6.1}$$

The term $-\mathrm{ord}(y^{p-1})$ corresponds to the ramification divisor of $W \to W/H$, the divisor defined by the Jacobian ideal, or to the relative canonical divisor of the proper birational map $\mathcal{W} \to W/H$. If $D_{\mathbf{d}} \geq 2$, then $W \to W/H$ is étale in codimension one and has no ramification divisor, but W/H acquires singularities as compensation. Therefore the corresponding formula in that case is

$$\int_C \mathbb{L}^{F+\mathcal{J}} \, d\mu_{W/H} = \int_{\phi^{-1}(C)} \mathbb{L}^{F\circ\phi-\mathfrak{s}} \, d\mu_{\mathcal{W}}. \tag{6.2}$$

When $C = J_\infty(W/H)$ and $F \equiv 0$, then the right hand side becomes

$$\int_{\mathcal{J}_\infty \mathcal{W}} \mathbb{L}^{-\mathfrak{s}} \, d\mu_{\mathcal{W}} = \int_{\Delta_H} \mathbb{L}^{-\mathrm{sht}'} \, d\mu_H.$$

We show a similar formula in the case of G. We first introduce a counterpart of $\mathcal{J}_\infty \mathcal{W}$. For $f \in \Delta_G$, let $\operatorname{Spec} K_f \to \operatorname{Spec} k((t))$ be the corresponding G-torsor. The underlying topological space of $\operatorname{Spec} K_f$ is always a singleton.

Lemma 6.1. *If $f \neq 0$, then K_f is reduced, equivalently, K_f is a field.*

Proof. The K_f is a finite extension of $k((t))$ of degree p. If K_f is non-reduced, then the associated reduced ring $(K_f)_{\mathrm{red}}$ is an extension of degree one, hence the natural map $k((t)) \to (K_f)_{\mathrm{red}}$ is an isomorphism. This means that the G-torsor $\operatorname{Spec} K_f \to \operatorname{Spec} k((t))$ admits a section, hence it is a trivial torsor and $f = 0$. □

For $f \neq 0$, let $\operatorname{Spec} \mathcal{O}_f$ be the normalization of $\operatorname{Spec} k[[t]]$ in $\operatorname{Spec} K_f$. For $f = 0$, we define $\mathcal{O}_f := k[[t]][z]/(z^p)$. We say that a morphism $\operatorname{Spec} \mathcal{O}_f \to V$ is G-equivariant if the composition map $\operatorname{Spec} K_f \to \operatorname{Spec} \mathcal{O}_f \to V$ is G-equivariant.

Note that the G-action on $\operatorname{Spec} K_f$ does not generally extend to $\operatorname{Spec} \mathcal{O}_f$, which is the reason that we define G-equivariant morphisms $\operatorname{Spec} \mathcal{O}_f \to V$ in this way. Let $\operatorname{Hom}^G(\operatorname{Spec} K_f, V)$ and $\operatorname{Hom}^G(\operatorname{Spec} \mathcal{O}_f, V)$ be the set of G-equivariant morphisms $\operatorname{Spec} K_f \to V$ and $\operatorname{Spec} \mathcal{O}_f \to V$ respectively. Regarding $\operatorname{Hom}(\operatorname{Spec} \mathcal{O}_f, V)$ as a subset of $\operatorname{Hom}(\operatorname{Spec} K_f, V)$, we have

$$\operatorname{Hom}^G(\operatorname{Spec} \mathcal{O}_f, V) = \operatorname{Hom}(\operatorname{Spec} \mathcal{O}_f, V) \cap \operatorname{Hom}^G(\operatorname{Spec} K_f, V).$$

We then put

$$\mathcal{J}_\infty \mathcal{V} := \bigsqcup_{f \in \Delta_G} \operatorname{Hom}^G(\operatorname{Spec} \mathcal{O}_f, V).$$

We now give an explicit description of this set. Let D denote the derivation on $k[x, y]$ given by $D(y) = x$, $D(x) = 0$, which corresponds to the G-action on $V = \operatorname{Spec} k[x, y]$. The G-action on V is given by the coaction:

$$\begin{aligned} \theta \colon k[x, y] &\to k[x, y][\epsilon] \\ x &\mapsto x \\ y &\mapsto y + x\epsilon \end{aligned}$$

The G-action on $\operatorname{Spec} K_f$ with $K_f = k((t))[z]/(z^p - f)$ is given by:

$$\begin{aligned} \psi_f \colon K_f &\to K_f[\epsilon] \\ z &\mapsto z + \epsilon \end{aligned}$$

An element of K_f is uniquely written as $\sum_{i=0}^{p-1} a_i z^i$, $a_i \in k((t))$ and a map $\gamma \colon \operatorname{Spec} K_f \to V$ is uniquely determined by two elements $\gamma^*(x) = \sum_{i=0}^{p-1} a_i z^i$ and $\gamma^*(y) = \sum_{i=0}^{p-1} b_i z^i$ of K_f. The map γ is G-equivariant if and only if

$$\psi_f(\gamma^*(x)) = (\gamma^* \otimes \mathrm{id}_{k[\epsilon]})(\theta(x)),\ \psi_f(\gamma^*(y)) = (\gamma^* \otimes \mathrm{id}_{k[\epsilon]})(\theta(y)).$$

The left equality is explicitly written as

$$\sum_i a_i (z + \epsilon)^i = \sum_i a_i z^i,$$

which is equivalent to saying that $a_i = 0$ for $i > 0$. The right equality then says that

$$\sum_i b_i (z + \epsilon)^i = \sum_i b_i z^i + \sum_i a_i z^i \epsilon.$$

This is equivalent to requiring $b_1 = a_0$ and $b_i = 0$ for $i > 1$. As a consequence, we can identify $\operatorname{Hom}^G(\operatorname{Spec} K_f, V)$ with

$$\{(a, b + az) \in K_f^2 \mid a, b \in k((t))\} \cong k((t))^2.$$

For $f \neq 0$, if we extend the order function on $k((t))$ to K_f as a valuation, then $\operatorname{ord}(z) = \frac{\operatorname{ord}(f)}{p}$. Therefore, with the above identification, $(a, b+az) \in \operatorname{Hom}^G(\operatorname{Spec} K_f, V)$ lies in $\operatorname{Hom}^G(\operatorname{Spec} \mathcal{O}_f, V)$ if and only if $\operatorname{ord}(a) \geq 0$ and $\operatorname{ord}(b+az) \geq 0$. Since $\operatorname{ord}(f)$ is coprime with p, we have $\mathbb{Z} \ni \operatorname{ord}(b) \neq \operatorname{ord}(az) \notin \mathbb{Z}$ and therefore

$$\operatorname{ord}(b + az) = \min\{\operatorname{ord}(b), \operatorname{ord}(az)\}$$

Thus the two conditions translate into $\operatorname{ord}(a) \geq \max\{0, \lceil -\operatorname{ord}(f)/p \rceil\}$ and $\operatorname{ord}(b) \geq 0$. In conclusion, for every f, the set $\operatorname{Hom}^G(\operatorname{Spec} \mathcal{O}_f, V)$ is identified with the following subset of $\mathcal{O}_f^2$,

$$\{(a, b+az) \in \mathcal{O}_f^2 \mid a \in t^{s_f} \cdot k[[t]],\ b \in k[[t]]\} \quad (s_f := \max\{0, \lceil -\operatorname{ord}(f)/p \rceil\} = \operatorname{sht}'(f) + 2).$$

We then identify $\mathcal{J}_\infty \mathcal{V}$ with

$$\bigsqcup_{f\in\Delta_G} t^{s_f}k[[t]] \oplus k[[t]]$$

and write its elements as triples (f, a, b). For $m \in \mathbb{Z}_{\geq 0}$, the image of $\mathrm{Hom}^G(\mathrm{Spec}\,\mathcal{O}_f, V)$ under the natural map $\mathcal{O}_f^2 \to \mathcal{O}_f^2/t^{m+1}\mathcal{O}_f^2$, which we denote by $\mathrm{Hom}^G(\mathrm{Spec}\,\mathcal{O}_f, V)_m$, coincides with the image of the injective map

$$\frac{t^{s_f}k[[t]]}{t^{s_f+m+1}k[[t]]} \oplus \frac{k[[t]]}{t^{m+1}k[[t]]} \longrightarrow \mathcal{O}_f^2/t^{m+1}\mathcal{O}_f^2,\ (a, b) \mapsto (a, b + az).$$

We define a motivic measure $\mu_\mathcal{V}$ on $\mathcal{J}_\infty\mathcal{V}$ as follows. For $m, n \in \mathbb{Z}_{\geq 0}$, let

$$\mathcal{J}_{m,n}\mathcal{V} := \bigsqcup_{f\in\Delta_{G,n}} \frac{t^{s_f}k[[t]]}{t^{s_f+m+1}k[[t]]} \oplus \frac{k[[t]]}{t^{m+1}k[[t]]}.$$

Here $s_f = s_g$ where $g \in \Delta_G$ is any lift of $f \in \Delta_{G,n}$. For integers n, j with $np \geq j$, let $\Delta_{G,n}^{\geq j} \subset \Delta_{G,n}$ be the subspace of $f \in \Delta_{G,n}$ with $\mathrm{ord}(f) \geq j$. This is an affine space of finite dimension and $\Delta_{G,n}$ is the union of $\Delta_{G,n}^{\geq j}$, $j \leq np$. This filtration also allows to write $\mathcal{J}_{m,n}\mathcal{V}$ as an increasing union of affine spaces, so that the notion of constructible subsets and their measure is well defined. For $m' \geq m$ and $n' \geq n$, the natural map $\mathcal{J}_{m',n'}\mathcal{V} \to \mathcal{J}_{m,n}\mathcal{V}$ is a trivial $\mathbb{A}_k^{2(m'-m)+(p-1)(n'-n)}$-bundle. Let $\pi_{m,n} \colon \mathcal{J}_\infty\mathcal{V} \to \mathcal{J}_{m,n}\mathcal{V}$ be the natural map. We say that a subset $C \subset \mathcal{J}_\infty\mathcal{V}$ is a *cylinder of level* (m, n) if $C = \pi_{m,n}^{-1}\pi_{m,n}(C)$ and $\pi_{m,n}(C)$ is a constructible subset of $\mathcal{J}_{m,n}\mathcal{V}$. Then we define the measure $\mu_\mathcal{V}(C)$ of C as

$$\mu_\mathcal{V}(C) := [\pi_{m,n}(C)]\mathbb{L}^{-2m-(p-1)n}.$$

We can further extend this measure $\mu_\mathcal{V}$ to measurable subsets, following [3, Appendix]. A subset $C \subset \mathcal{J}_\infty\mathcal{V}$ is *measurable* if there exists a sequence of cylinders $C_1, C_2, \dots$ approximating C (which means that there exists another sequence $B_1, B_2, \dots$ of cylinders such that $\lim_{i\to\infty} \mu_\mathcal{V}(B_i) = 0$ and for each i, the symmetric difference $C \triangle C_i = (C \cup C_i) \setminus (C \cap C_i)$ is contained in B_i). For a measurable subset C, we define $\mu_\mathcal{V}(C) := \lim_{i\to\infty} \mu_\mathcal{V}(C_i)$. A function $f \colon C \to \mathbb{Z}$ on a subset $C \subset \mathcal{J}_\infty\mathcal{V}$ is said to be *measurable* if all fibers $f^{-1}(n)$ are measurable. The integral $\int_C \mathbb{L}^f\, d\mu_\mathcal{V}$ is then defined to be $\sum_{n\in\mathbb{Z}}[f^{-1}(n)]\mathbb{L}^n$ in $\hat{\mathcal{M}}'$, provided that this infinite sum converges.

The quotient variety V/G has the coordinate ring $k[x, y^p]$. The arc space $J_\infty(V/G)$ of V/G is identified with $k[[t]]^2$ by looking at the images of x and y^p in $k[[t]]$. Similarly the m-th jet scheme $J_m(V/G)$ is identified with $\left(k[[t]]/(t^{m+1})\right)^2$. Given an element of $\mathcal{J}_\infty\mathcal{V}$ regarded as a G-equivariant morphism $\mathrm{Spec}\,\mathcal{O}_f \to V$, taking the G-quotient of the induced morphism $\mathrm{Spec}\,K_f \to V$, we obtain a morphism $\mathrm{Spec}\,k((t)) \to V/G$. We easily see that this morphism extends to a morphism $\mathrm{Spec}\,k[[t]] \to V/G$. Thus we obtain a map $\psi \colon \mathcal{J}_\infty\mathcal{V} \to J_\infty(V/G)$. In concrete terms, the map sends (f, a, b) to $(a, b^p + fa^p)$. For $n \geq m \geq 0$, the map ψ induces map

$$\psi_{m,n} \colon \mathcal{J}_{m,n}\mathcal{V} \to J_m(V/G),\ (f, a, b) \mapsto (a, b^p + fa^p).$$

Let us take an element

$$(\alpha, \beta) \in \left(\frac{k[[t]]}{t^{m+1}k[[t]]}\right)^{\oplus 2} = J_m(V/G).$$

We will describe the fiber $\psi_{m,n}^{-1}((\alpha,\beta))$. Namely we will describe the set of triples (f,a,b) with $f\in\Delta_{G,n}$, $a\in t^{s_f}k[[t]]/t^{s_f+m+1}k[[t]]$, $b\in k[[t]]/t^{m+1}k[[t]]$ such that $a=\alpha$ and $b^p+fa^p=\beta$ in $k[[t]]/t^{m+1}k[[t]]$. Let us write $a=\sum_{i\le s_f+m}a_it^i$, $b=\sum_{i\le m}b_it^i$, $f=\sum_{i\le np-1}f_it^i$. The equality $a=\alpha$ determines a_i, $i\le m$, requires that $s_f\le \mathrm{ord}(\alpha)$ and put no other constraint on a_i, $i>m$, b_i or f_i. For the equality $b^p+fa^p=\beta$, we note that b^p (resp. fa^p) has only terms of degrees divisible (resp. not divisible) by p. Therefore this equality determines b_i, $i\le\lfloor m/p\rfloor$. If a is fixed and $m\ge p\cdot\mathrm{ord}(a)=p\cdot\mathrm{ord}(\alpha)$, then the same equation determines f_i, $i\le m-p\cdot\mathrm{ord}(a)$, but put no more constraint on a_i, b_i, f_i. In this case, since the resulting f is such that fa^p has neither term of negative degree nor term of degree divisible by p, it does follow that $f\in\Delta_{G,n}$ and that $s_f\le\mathrm{ord}(\alpha)$. Notice moreover that, if β' is the subsum of β of degree coprime with p, then s_f is a function of $\mathrm{ord}(\beta')$, so that, in particular, the number s_f does not depends of the choice of (f,a,b) over (α,β). For simplicity, suppose $m=m'p$ for some $m'\in\mathbb{N}$. As a consequence of the above computation, if $\alpha\ne0$ and $m'\ge\mathrm{ord}(\alpha)$, then $\psi_{m,n}^{-1}((\alpha,\beta))$ is the affine space of dimension

$$\begin{aligned}&\overbrace{\{(s_f+m)-m\}}^{\text{no. of free }a_i}+\overbrace{(m-\lfloor m/p\rfloor)}^{\text{no. of free }b_i}+\overbrace{\{n-(m'-\mathrm{ord}(a))\}(p-1)}^{\text{no. of free }f_i}\\&=s_f+(p-1)n+(p-1)\mathrm{ord}(a).\end{aligned}$$

We define functions

$$s\colon \mathcal{J}_\infty\mathcal{V}\to\mathbb{Z},\ (f,a,b)\mapsto s_f=\mathrm{sht}'(f)+2,$$
$$\mathrm{ord}(x)\colon \mathcal{J}_\infty\mathcal{V}\to\mathbb{Z}\sqcup\{\infty\},\ (f,a,b)\mapsto\mathrm{ord}(a).$$

From the above argument, s is the composition of $\psi\colon\mathcal{J}_\infty\mathcal{V}\to J_\infty(V/G)$ and a function $s'\colon J_\infty(V/G)\to\mathbb{Z}$, the latter having cylindrical fibers. The function $\mathrm{ord}(x)$ on $\mathcal{J}_\infty\mathcal{V}$ also factors through $\mathrm{ord}(x)\colon J_\infty(V/G)\to\mathbb{Z}\cup\{\infty\}$, whose fibers are also cylinders except that $\mathrm{ord}(x)^{-1}(\infty)$ is a measurable subset of measure zero. Let $C\subset J_\infty(V/G)$ be a cylinder of level m. The inverse image $\psi^{-1}(C)$ is a cylinder of level (m,m) and

$$\pi_{m,m}(\psi^{-1}(C))=\psi_{m,m}^{-1}(\pi_m(C)).$$

If s and $\mathrm{ord}(x)$ take constant values s_0 and r on $\psi^{-1}(C)$, then

$$\mu_{V/G}(C)=\mathbb{L}^{-s_0-(p-1)r}\mu_{\mathcal{V}}(\psi^{-1}(C)).$$

By a standard formal argument on measurable subsets (for instance, see [10, Proof of Th. 5.20]), this equality is valid also when C is a measurable subset. Subdividing a given measurable subset $C\subset J_\infty(V/G)$, we can reduce to the case where s and $\mathrm{ord}(x)$ are constant. These arguments lead to the change of variables formula:

Theorem 6.2. *Let $C\subset J_\infty(V/G)$ be a subset and $F\colon C\longrightarrow\mathbb{Z}$ be a measurable function. Then*

$$\int_C\mathbb{L}^F\,d\mu_{V/G}=\int_{\psi^{-1}(C)}\mathbb{L}^{F\circ\psi-s-(p-1)\mathrm{ord}(x)}\,d\mu_{\mathcal{V}}.$$

In particular,

$$\mathbb{L}^2=M_{\mathrm{st}}(V/G)=\int_{J_\infty(V/G)}1\,d\mu_{V/G}=\int_{\mathcal{J}_\infty\mathcal{V}}\mathbb{L}^{-s-(p-1)\mathrm{ord}(x)}\,d\mu_{\mathcal{V}}.$$

In the last line $M_{\mathrm{st}}(V/G) = \int_{J_\infty(V/G)} 1\, d\mu_{V/G}$ follows from definition and the fact that V/G is smooth, while $\int_{J_\infty(V/G)} 1\, d\mu_{V/G} = \mathbb{L}^2$ from the fact that $J_0(V/G) = V/G = \mathbb{A}^2_k$.

We describe here an alternative way to check the equality

$$\int_{\mathcal{J}_\infty \mathcal{V}} \mathbb{L}^{-s-(p-1)\mathrm{ord}(x)}\, d\mu_{\mathcal{V}} = \mathbb{L}^2.$$

The set

$$C_{\geq 0,i} := \{(f,a,b) \in \mathcal{J}_\infty \mathcal{V} \mid \mathrm{ord}(f) \geq 0,\, \mathrm{ord}(a) = i\}$$

is a cylinder of level $(i,0)$ with $\pi_{i,0}(C_{\geq 0,i}) \cong \mathbb{G}_m \times \mathbb{A}^{i+1}_k$. Therefore

$$\mu_{\mathcal{V}}(C_{\geq 0,i}) = (\mathbb{L}-1)\mathbb{L}^{i+1}\mathbb{L}^{-2i} = (\mathbb{L}-1)\mathbb{L}^{-i+1}$$

Their disjoint union is $C_{\geq 0} := \{(f,a,b) \mid \mathrm{ord}(f) \geq 0\}$ and

$$\begin{aligned}\int_{C_{\geq 0}} \mathbb{L}^{-s-(p-1)\mathrm{ord}(x)}\, d\mu_{\mathcal{V}} &= \sum_{i\geq 0} \mu_{\mathcal{V}}(C_{0,i})\mathbb{L}^{-(p-1)i} \\ &= \frac{\mathbb{L}^2-\mathbb{L}}{1-\mathbb{L}^{-p}}.\end{aligned}$$

For $j = -(pd+e) < 0$ with $d \in \mathbb{N}$ and $1 \leq e \leq p-1$ and for $i \in \mathbb{N}$, let $C_{j,i} := \{(f,a,b) \in \mathcal{J}_\infty \mathcal{V} \mid \mathrm{ord}(f) = j,\, \mathrm{ord}(a) = s_f + i\}$. This set is a cylinder of level $(i,0)$ such that

$$\pi_{i,0}(C_{j,i}) \cong \overbrace{\mathbb{G}_m \times \mathbb{A}^{d(p-1)+e-1}_k}^{f} \times \overbrace{\mathbb{G}_m}^{a} \times \overbrace{\mathbb{A}^{i+1}_k}^{b}.$$

Thus

$$\mu_{\mathcal{V}}(C_{j,i}) = (\mathbb{L}-1)^2\mathbb{L}^{-i+d(p-1)+e}.$$

The disjoint union of all the $C_{j,i}$ is $C_{<0} = \{(f,a,b) \mid \mathrm{ord}(f) < 0\}$. Then

$$\begin{aligned}\int_{C_{<0}} \mathbb{L}^{-s-(p-1)(x)}\, d\mu_{\mathcal{V}} &= \sum_{1\leq e\leq p-1}\sum_{d\in\mathbb{N}}\sum_{i\in\mathbb{N}} (\mathbb{L}-1)^2\mathbb{L}^{-i+d(p-1)+e} \times \mathbb{L}^{-(d+1)-(p-1)(d+1+i)} \\ &= \sum_{1\leq e\leq p-1}\sum_{d\in\mathbb{N}}\sum_{i\in\mathbb{N}} (\mathbb{L}-1)^2\mathbb{L}^{-pi+e-d-p} \\ &= (\mathbb{L}-1)^2\mathbb{L}^{-p}\frac{\mathbb{L}^1+\cdots+\mathbb{L}^{p-1}}{(1-\mathbb{L}^{-p})(1-\mathbb{L}^{-1})} \\ &= \frac{\mathbb{L}-\mathbb{L}^{2-p}}{1-\mathbb{L}^{-p}}.\end{aligned}$$

It follows that

$$\int_{\mathcal{J}_\infty \mathcal{V}} \mathbb{L}^{-s-(p-1)\mathrm{ord}(x)}\, d\mu_{\mathcal{V}} = \frac{\mathbb{L}^2-\mathbb{L}}{1-\mathbb{L}^{-p}} + \frac{\mathbb{L}-\mathbb{L}^{2-p}}{1-\mathbb{L}^{-p}} = \mathbb{L}^2.$$

Remark 6.3. It is natural to see the function $(p-1)\mathrm{ord}(x)$ in Theorem 6.2 as a counterpart of the function $\mathrm{ord}(y^{p-1}) = (p-1)\mathrm{ord}(y)$ in the change of variables formula (6.1) for the case of H. The latter is the order function associated to the Jacobian ideal $(y^{p-1}) \subset k[x,y]$ of the map $W \to W/H$. It is a natural problem, how to derive the ideal (x^{p-1}) as the "Jacobian ideal" of $V \to V/G$, a map not generically étale.

Appendix A. Representation theory of α_p

In this appendix we recall the representation theory of the group scheme α_p over $\mathbb{F}_p$.

Given an $\mathbb{F}_p$-algebra A we denote by $\mathrm{Mod}^{\alpha_p} A$ the category of A-modules with an action of $\alpha_p \times A$, or, equivalently, a coaction of the Hopf algebra $A[\alpha_p] = A[\varepsilon]$. We introduce also the category $\mathrm{Mod}^{nil} A$ of pairs (M, ξ) where M is an A-module and $\xi\colon M \longrightarrow M$ is an A-linear map which is p-nilpotent, that is $\xi^p = 0$. Given $(M, \xi) \in \mathrm{Mod}^{nil} A$ we define

$$\exp(\xi\varepsilon) = \sum_{i=0}^{p-1} \frac{\xi^i \varepsilon^i}{i!} : M \longrightarrow M \otimes \mathbb{F}_p[\varepsilon]$$

Proposition A.1. *The functor*

$$\mathrm{Mod}^{nil}(A) \longrightarrow \mathrm{Mod}^{\alpha_p}(A),\ (M, \xi) \longmapsto (M, \exp(\xi\varepsilon))$$

is well defined and an equivalence of categories. Moreover for (M, ξ), $(N, \eta) \in \mathrm{Mod}^{nil}(A)$ we have $M^{\alpha_p} = \mathrm{Ker}(\xi)$ and that

$$\xi \otimes \mathrm{id}_M + \mathrm{id}_N \otimes \eta\colon M \otimes N \longrightarrow M \otimes N$$

corresponds to the tensor product of α_p-modules.

If B is an A-algebra and $(B, \xi) \in \mathrm{Mod}^{nil}(A)$ then α_p acts on the A-algebra B, that is $A \longrightarrow B$ and $B \otimes_A B \longrightarrow B$ are α_p-equivariant, if and only if $\xi\colon B \longrightarrow B$ is an A-derivation.

Proof. Let M be an A-module and $\phi\colon M \longrightarrow M \otimes k[\varepsilon]$ be an A-linear map. The map ϕ can be written as

$$\phi = \sum_{i=0}^{p-1} \phi_i \varepsilon^i \text{ for } A\text{-linear maps } \phi_i\colon M \longrightarrow M$$

The map ϕ must satisfy the following two conditions in order to be an α_p-action: $(\mathrm{id}_M \otimes z) \circ \phi = \mathrm{id}_M\colon M \longrightarrow M$, where $z\colon \mathbb{F}_p[\varepsilon] \longrightarrow \mathbb{F}_p$, $z(\varepsilon) = 0$ is the 0-section and $(\mathrm{id}_M \otimes \Delta) \circ \phi = (\phi \otimes \mathrm{id}_{\mathbb{F}_p[\epsilon]}) \circ \phi\colon M \longrightarrow M \otimes \mathbb{F}_p[\varepsilon] \otimes \mathbb{F}_p[\varepsilon]$, where $\Delta\colon \mathbb{F}_p[\varepsilon] \longrightarrow \mathbb{F}_p[\varepsilon] \otimes \mathbb{F}_p[\varepsilon]$, $\Delta(\varepsilon) = \varepsilon \otimes 1 + 1 \otimes \varepsilon$ is the comultiplication. Those conditions translate into

$$\phi_0 = \mathrm{id}_M \text{ and } \phi_i \phi_j = \binom{i+j}{i} \phi_{i+j} \text{ for } 0 \leq i, j < p$$

and into $\phi_1^p = 0$, $\phi = \exp(\phi_1 \varepsilon)$. This easily prove the equivalence in the statement.

The trivial α_p-action on A corresponds to the nilpotent endomorphism $A \xrightarrow{0} A$. Thus

$$M^{\alpha_p} = \mathrm{Hom}^{\alpha_p}(A, M) = \mathrm{Hom}_{\mathrm{Mod}^{nil}(A)}((A, 0), (M, \xi)) = \mathrm{Ker}(\xi)$$

The claim about the tensor product follows from a direct check.

Consider now the last statement. The map $\iota\colon A \longrightarrow B$ is α_p-equivariant if ι is compatible with the nilpotent endomorphisms $A \xrightarrow{0} A$ and ξ if and only if $\xi(\iota(a)) = 0$ for $a \in A$. From the assertion of the tensor product, the map $B \otimes_A B \longrightarrow B$ is α_p-equivariant if and only if this map is compatible with $\xi\colon B \longrightarrow B$ and $\xi \otimes \mathrm{id}_B + \mathrm{id}_B \otimes \xi\colon B \otimes_A B \to B \otimes_A B$ if and only if ξ satisfies the Leibniz rule. This ends the proof. □

Example A.2. Let $(M, \xi) \in \mathrm{Mod}^{nil} A$. Then α_p acts on the A-algebra $\mathrm{Sym}(M)$ and the corresponding p-nilpotent endomorphism $\xi_* \colon \mathrm{Sym}(M) \longrightarrow \mathrm{Sym}(M)$ is the unique A-derivation such that $(\xi_*)_{|M} = \xi$.

In particular the corresponding p-nilpotent endomorphism $\xi_n \colon \mathrm{Sym}^n M \longrightarrow \mathrm{Sym}^n M$ is given by

$$\xi_n(m_1 \cdots m_n) = \xi(m_1)m_2 \cdots m_n + \cdots + m_1 \cdots m_{n-1}\xi(m_n)$$

Example A.3. Assume that $A = k$ is a field. Then any p-nilpotent endomorphism $\xi \colon k^n \longrightarrow k^n$ can be put in Jordan form and, in this case, this just means that all blocks have 0 diagonal and have size at most p. It follows that, up to isomorphisms, the α_p-represention over k correspond bijectively to sequences $\mathbf{d} = (d_1, \ldots, d_l)$ with $1 \leq d_i \leq p$, $d_i \geq d_{i+1}$ and $l \in \mathbb{N}$.

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